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Multi-mode approximations to wave scattering by an uneven bed

By P. G. CHAMBERLAIN AND D. PORTER

Department of Mathematics, The University of Reading, PO Box 220, Whiteknights, Reading, RG6 6AX, UK

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Approximations to the scattering of linear surface gravity waves on water of varying quiescent depth are investigated by means of a variational approach. Previous authors have used wave modes associated with the constant depth case to approximate the velocity potential, leading to a system of coupled differential equations. Here it is shown that a transformation of the dependent variables results in a much simplified differential equation system which in turn leads to a new multi-mode 'mild-slope' approximation. Further, the effect of adding a bed mode is examined and clarified. A systematic analytic method is presented for evaluating inner products that arise and numerical experiments for two-dimensional scattering are used to examine the performance of the new approximations.

1. Introduction

The purpose of this paper is to extend the investigation carried out by Porter (2003) for the mild-slope equation to a multi-mode approximation of surface gravity wave scattering by topography. The aim is to derive a system of relatively simple differential equations that can produce the solution of the full linear scattering problem arbitrarily closely.

The mild-slope equation of Berkhoff (1972, 1976) and Smith & Sprinks (1975) and the related extended mild-slope equation (Kirby 1986) and modified mild-slope equation (Chamberlain & Porter 1995) have been well documented. For example, Porter (2003) refers to other derivations and Athanassoulis & Belibassakis (1999) provide a comprehensive survey, including multi-mode approximations that concern us here. The essential point is that the mild-slope equation and its variants arise from approximating surface wave motions over topography by a single term in which the vertical coordinate arises only in a known eigenfunction that corresponds to propagating waves in water of constant depth. Vertical integration, essentially an application of Galerkin's method, or the use of a variational principle then removes the vertical coordinate completely and the dimension of the problem is thereby reduced. This simplification has been widely exploited in a range of problems, including wave trapping and near-trapping (Chamberlain & Porter 1996, 1999) and scattering by doubly modulated topography (Porter & Porter 2000).

However, the drawback in using the mild-slope approximation is that its accuracy is not known. The results obtained are likely to be reasonably good approximations if the topography is slowly varying, but even in this case the derivation does not lead to an error term that can be estimated. To obtain more reliable quantitative information, further terms must be included in the approximation. The natural extension to the basic single-term mild-slope approximation is to combine it with additional terms containing eigenfunctions corresponding to a number of evanescent modes. This idea was implemented by Massel (1993) using Galerkin's method and by Porter & Staziker (1995), who invoked a variational principle. It was shown that, by including N additional terms and increasing N until a desired accuracy has been reached, the solution of the unapproximated problem is obtained, to that accuracy. The removal of the vertical coordinate again simplifies the problem when the extended 'multi-mode' approximation is used but the modified mild-slope equation is replaced by $N+1$ coupled partial differential equations in the horizontal variables. Porter & Staziker (1995) also drew attention to the fact that discontinuities in the bed slope have a significant effect on the predicted wave heights, which had previously not been accounted for even in the mild-slope model.

Two further developments form the background to the present paper. Athanassoulis & Belibassakis (1999) showed that the multi-mode approach used by Massel (1993) and Porter & Staziker (1995) is flawed in the sense that a pointwise representation of the exact solution in the whole flow domain cannot be obtained in the limit $N \to \infty$ in which all of the eigenfunctions corresponding to evanescent modes are included. This limitation arises because the approximation cannot satisfy the exact bed condition as it is formed from eigenfunctions that do not, and as a result the approximation to the fluid flow near the bed will converge slowly. Athanassoulis & Belibassakis (1999) showed that the addition of a 'sloping-bottom' mode to the natural eigenfunction expansion set overcomes this deficiency and, inevitably, improves the convergence rate of the approximation. In practical terms, a smaller value of N is required to obtain a given accuracy when the extra sloping-bottom mode is included.

The second point to be made is that in a reappraisal of the basic one-term approximation, Porter (2003) showed that the introduction of a particular scaling both simplifies the modified mild-slope equation and removes the effect of bed slope discontinuities that are implicit in the established form. Further, a new version of the mild-slope equation emerged, simpler than that originally derived and yet containing all of the improvements provided by the more complicated modified mild-slope equation. Chamberlain & Porter (2005) have extended this approach to a two-layer fluid over uneven beds and obtained corresponding simplifications. We pursue the approach here and extend it to the case of multi-mode approximations for a single fluid, which we thereby are able to simplify. We also examine the effect of including a sloping-bottom, or bed, mode, but use this in a different way to Athanassoulis & Belibassakis (1999).

The plan of the paper is as follows. Section 2 comprises a formulation of the problem including a brief account of the corresponding variational approximation. In the following section, it is shown that the differential equation system that results from the standard approach can be simplified by means of a transformation, which is first applied to the natural-wave-mode approximating set; a bed slope term is then added. The significance of the transformation and of the bed mode is explored numerically in § 4, for which purpose we consider a class of scattering problems in two dimensions. This allows us to implement standard test problems used by previous authors. It also provides a definitive estimate of error for we may use results obtained by the method derived in Porter & Porter (2000), which solves the full linear problem to any accuracy. That method, which appears to obviate the need for approximation models, was developed to benchmark approximation methods rather than as a practical solution technique; it deals only with two-dimensional problems and is significantly less convenient to use than the approach described in the present paper, which applies in the three-dimensional case.

2. Formulation

Cartesian coordinates are used with the x and y axes lying in the mean free surface $z = 0$, where z is directed vertically upwards. The bed is at $z = -h$, $h(x, y)$ being a given continuous function with piecewise continuous first derivatives.

The usual assumptions of linearized theory and the removal of the harmonic time dependence $\exp(-\mathrm{i}\sigma t)$ lead to equations for the time-independent velocity potential $\phi(x, y, z)$ in the form

$$\left.\begin{aligned} \nabla^2\phi &= 0 \quad (-h < z < 0), \\ \phi_z - K\phi &= 0 \quad (z = 0), \\ \phi_z + \nabla_h h \cdot \nabla\phi &= 0 \quad (z = -h), \end{aligned}\right\} \tag{2.1}$$

where $K = \sigma^2/g$ and $\nabla_h = (\partial/\partial x, \partial/\partial y, 0)$; g is acceleration due to gravity. Radiation conditions to be applied as $x^2 + y^2 \to \infty$ are required to complete the specification of ϕ, but our immediate concern is with the set of equations (2.1). We note that the free-surface elevation is recovered from ϕ by

$$\zeta(x, y, t) = \mathrm{Re}\{\eta(x, y)\mathrm{e}^{-\mathrm{i}\sigma t}\}, \quad \eta(x, y) = -\mathrm{i}\sigma[\phi]_{z=0}/g.$$

We follow Porter & Staziker (1995) by using a variational principle to generate approximations to the solution of (2.1). Here we take the real-valued functional

$$L(\varphi) = \iint_{\mathscr{D}} \mathscr{L} \mathrm{d}x\, \mathrm{d}y, \quad 2\mathscr{L}(\varphi, \nabla\varphi) = K[|\varphi|^2]_{z=0} - \int_{-h}^{0} \nabla\varphi \cdot \overline{\nabla\varphi}\, \mathrm{d}z,$$

where $\mathscr{D}$ denotes a simply connected domain in the plane $z = 0$ with boundary $\mathscr{C}$. A bar is used to denote complex conjugation. The significance of this choice is that $\rho\mathscr{L}$, where ρ denotes the constant fluid density, is the difference between the potential energy and the kinetic energy of the vertical filament of fluid at (x, y), averaged over a period $2\pi/\sigma$. $\mathscr{L}$ is therefore proportional to the time-averaged Lagrangian density of the motion.

A straightforward calculation gives

$$\delta L = 2\mathrm{Re}\left\{\iint_{\mathscr{D}} \{[\delta\overline{\varphi}\,(K\varphi - \varphi_z)]_{z=0} + [\delta\overline{\varphi}\,(\varphi_z + \nabla_h h \cdot \nabla\varphi)]_{z=-h} + \int_{-h}^{0} \delta\overline{\varphi}\, \nabla^2\varphi\, \mathrm{d}z\}\, \mathrm{d}x\, \mathrm{d}y - \int_{\mathscr{C}} \int_{-h}^{0} \delta\overline{\varphi}\, \boldsymbol{n}_c \cdot \nabla_h \varphi\, \mathrm{d}z\, \mathrm{d}c\right\}, \tag{2.2}$$

where $\boldsymbol{n}_c$ is the outward unit normal on $\mathscr{C}$. Since our first objective is to approximate the dependence of ϕ on z, we may consider variations $\delta\varphi$ that vanish on the lateral boundary $\mathscr{C} \times [-h, 0]$; the radiation conditions will be reinstated for the approximate solution later. It follows from (2.2) that L is stationary at $\varphi = \phi$ if, and only if, ϕ satisfies the conditions (2.1). Thus we can select a function φ of a particular form to approximate ϕ and optimize this approximation by enforcing $\delta L = 0$.

In particular, we can choose φ so as to remove the coordinate z from the resulting approximation to the boundary-value problem. To achieve this we set

$$\phi \approx \varphi = \sum_{n=M}^{N} \phi_n(x, y) Z_n(h(x, y), z), \tag{2.3}$$

in which the real-valued functions $Z_M, \ldots, Z_N$ are assumed to be given. We take $N \geqslant 0$ and $M = -1$ or $M = 0$; the reason for this notation will become clear shortly. We suppose that each Z_n satisfies the free-surface condition $Z_n' - KZ_n = 0$ on $z = 0$, where the dash denotes differentiation with respect to z, but otherwise we do not specify the expansion set at this stage.

To express the result of this approximation process concisely, it is convenient to use the inner product notation

$$(u, v) = \int_{-h}^{0} u(z)\overline{v(z)}\,\mathrm{d}z, \quad (u, u) = ||u||^2,$$

and a dot to denote differentiation with respect to h; thus $\dot{Z}_n = \partial Z_n/\partial h$, for example.

The effect of substituting the expression (2.3) for φ into L and imposing the stationary condition $\delta L = 0$ can be deduced by using (2.2). After some manipulation, we find that the functions ϕ_n must satisfy the coupled system of equations

$$\sum_{n=M}^{N} \big\{\nabla_h \cdot (Z_n, Z_m)\nabla_h \phi_n + \{(Z_m, \dot{Z}_n) - (Z_n, \dot{Z}_m)\}\nabla_h h \cdot \nabla_h \phi_n + \{(Z_m, Z_n'') + [Z_m Z_n']_{z=-h} + (Z_m, \dot{Z}_n)\nabla_h^2 h + ((Z_m, \dot{Z}_n)^{\bullet} - (\dot{Z}_m, \dot{Z}_n))(\nabla_h h)^2\}\phi_n\big\} = 0, \quad (2.4)$$

holding for $m = M, \ldots, N$. The approximation $\varphi \approx \phi$ is therefore determined by solving this system for $\phi_M, \ldots, \phi_N$, subject to appropriate radiation conditions.

However, the system applies only where h is differentiable. If we suppose that $\nabla_h h$ is discontinuous on a curve whose projection onto $z = 0$ is denoted by Γ, then it is easy to deduce from (2.4) that the term $\nabla_h^2 h$ induces the set of jump conditions

$$\sum_{n=M}^{N} \big\{(Z_m, Z_n)\langle \boldsymbol{n}_\Gamma \cdot \nabla_h \phi_n\rangle + (Z_m, \dot{Z}_n)\langle \boldsymbol{n}_\Gamma \cdot \nabla_h h\rangle \phi_n\big\} = 0, \quad (2.5)$$

for $m = M, \ldots, N$. Here we have used $\langle\ .\ \rangle$ to denote the jump in the included quantity across Γ and $\boldsymbol{n}_\Gamma$ is the normal to Γ defined in a consistent way. Thus, if h has a discontinuous slope at one or more locations in the given domain, (2.4) must be solved on subdomains, and the piecewise solutions related through (2.5). Porter & Staziker (1995) obtained the equivalent set of conditions to (2.5) directly from an amended form of a variational principle, and showed that it corresponds to conservation of mass. The same procedure can be carried out in the present formulation.

The simplest approximation of the type under consideration corresponds to $M = N = 0$ and

$$Z_0(h, z) = \mathrm{sech}(kh)\cosh k(z + h), \quad (2.6)$$

where the wavenumber $k = k(x, y)$ is the positive real root of the dispersion relation

$$K = k \tanh(kh) \quad (2.7)$$

with $h = h(x, y)$, the frequency parameter K being assumed fixed. In this case, (2.4) reduces to the modified mild-slope equation derived by Chamberlain & Porter (1995).

The natural extension to a higher-order approximation, introduced by Massel (1993) and Porter & Staziker (1995) supplements Z_0 with

$$Z_n(h, z) = \sec(k_n h)\cos k_n(z + h) \quad (n = 1, \ldots, N). \quad (2.8)$$

Here, k_n denote the positive roots of

$$k_n \tan(k_n h) = -K, \tag{2.9}$$

arranged so that $k_1 < k_2 < \ldots$. It is convenient to write $k_0 = -\mathrm{i}k$, to subsume Z_0 into the definition (2.8). In the limit $N \to \infty$, $Z_0, Z_1, \ldots$ is the complete orthogonal set of vertical eigenfunctions obtained by separating the variables in (2.1) for a horizontal bed; Z_0 corresponds to a propagating mode and Z_n for $n \geqslant 1$ to evanescent modes.

However, as remarked in § 1, Athanassoulis & Belibassakis (1999) noted that the exact solution $\phi(x, y, z)$ cannot be represented pointwise as an expansion in the complete set Z_n $(n = 0, 1, \ldots)$ for $-h \leqslant z \leqslant 0$; this expansion will not converge to a function ϕ satisfying the bed condition at $z = -h$ because $Z_n' = 0$ there, for every $n = 0, 1, \ldots$. Although we are considering only a truncated expansion set here and can choose it however we wish, this observation is nevertheless significant in the approximation context for, as Athanassoulis & Belibassakis showed, the convergence rate is improved by extending the expansion set to include a term Z_{-1} such that $Z_{-1}' \neq 0$ at $z = -h$. (Clearly, $M = -1$ has been incorporated in the formulation to include a sloping-bottom mode.) This is an example of the general property of variational approximations, that the dimension of the expansion set will be reduced the more closely the set models the properties of the exact solution. Athanassoulis & Belibassakis (1999) extended the set Z_n $(n = 0, 1, \ldots, N)$ by additional function

$$Z_{-1}(h, z) = h\{(z/h)^3 + (z/h)^2\}, \tag{2.10}$$

which satisfies the free-surface condition and is such that $Z_{-1}' = 1$ on $z = -h$. They reported, however, that different choices of Z_{-1} having these properties lead to the same numerical results.

3. A transformation

One contribution of the present work is the derivation of a new multi-mode approximation based on the existing framework derived above. As noted in § 1 this approach is motivated by a reappraisal of the modified mild-slope equation by Porter (2003), which resulted in a more transparent and concise version of the equation. Here, we seek to simplify the system (2.3) in a corresponding way.

This is achieved by seeking a different expansion set to that used in (2.3). More precisely, we express (2.3) in a different form by retaining the notionally prescribed functions Z_n and introducing a new set $W_M(h, z), \ldots, W_N(h, z)$ defined in terms of $Z_M, \ldots, Z_N$.

To implement the process concisely, we first note that the approximation (2.3) may be written as

$$\phi \approx \varphi = \boldsymbol{\Phi}^T \boldsymbol{Z}, \quad \boldsymbol{\Phi} = (\phi_M, \ldots, \phi_N)^T, \boldsymbol{Z} = (Z_M, \ldots, Z_N)^T. \tag{3.1}$$

If we now introduce the transformation

$$\boldsymbol{\Phi} = S(h)\boldsymbol{\Psi}, \quad \boldsymbol{W} = S^T(h)\boldsymbol{Z}, \quad \boldsymbol{\Psi} = (\psi_M, \ldots, \psi_N)^T, \quad \boldsymbol{W} = (W_M, \ldots, W_N)^T, \tag{3.2}$$

in which the non-singular $(N - M + 1) \times (N - M + 1)$ matrix $S(h)$ is to be determined, the approximation takes the form

$$\phi \approx \varphi = \sum_{n=M}^{N} \psi_n(x, y) W_n(h(x, y), z). \tag{3.3}$$

Noting that the property $W_n' - KW_n = 0$ on $z = 0$ is inherited from Z_n through (3.2), the use of (3.3) in the variational principle $\delta L = 0$ obviously results in a system of the form (2.4) with ϕ_n replaced by ψ_n and Z_n by W_n. Thus

$$\sum_{n=M}^{N} \{\nabla_h \cdot (W_n, W_m)\nabla_h \psi_n + \{(W_m, \dot{W}_n) - (W_n, \dot{W}_m)\}\nabla_h h \cdot \nabla_h \psi_n + \{(W_m, W_n'') + [W_m W_n']_{z=-h} + (W_m, \dot{W}_n)\nabla_h^2 h + ((W_m, \dot{W}_n)^{\boldsymbol{\cdot}} - (\dot{W}_m, \dot{W}_n))(\nabla_h h)^2\}\psi_n\} = 0. \quad (3.4)$$

At this stage, we can choose $S = S(h)$ to be any invertible matrix, but we will restrict our attention to those transformations which remove the term involving $\nabla_h^2 h$. We therefore require that

$$(W_m, \dot{W}_n) = 0, \quad (m, n = M, \ldots, N). \quad (3.5)$$

It follows that (3.4) simplifies to

$$\sum_{n=M}^{N} \{\nabla_h \cdot A_{mn} \nabla_h \psi_n + \{B_{mn} - C_{mn}(\nabla_h h)^2\}\psi_n\} = 0, \quad (3.6)$$

for $m = M, \ldots, N$, in which

$$A_{mn} = (W_m, W_n), \quad B_{mn} = (W_m, W_n'') + [W_m W_n']_{z=-h}, \quad C_{mn} = (\dot{W}_m, \dot{W}_n).$$

The approximation (3.3) is completed by solving for $W_M, \ldots, W_N$, that is, by determining the transformation matrix $S(h)$. Now (3.2) implies that

$$\boldsymbol{W}\dot{\boldsymbol{W}}^T = S^T(\boldsymbol{Z}\dot{\boldsymbol{Z}}^T S + \boldsymbol{Z}\boldsymbol{Z}^T \dot{S}),$$

the matrix on the left-hand side having $m - M + 1, n - M + 1$ component $W_m \dot{W}_n$ $(m, n = M, \ldots, N)$. Integrating with respect to z from $-h$ to 0 and enforcing (3.5), we deduce that S must satisfy

$$\mathscr{A}\dot{S} + \mathscr{D}S = 0, \quad \mathscr{A}_{mn} = (Z_m, Z_n), \quad \mathscr{D}_{mn} = (Z_m, \dot{Z}_n), \quad (3.7)$$

$\mathscr{A}$ and $\mathscr{D}$ being $(N - M + 1) \times (N - M + 1)$ matrices.

The obvious advantages of the system (3.6) over (2.4) are its relative simplicity and the continuity of the functions $\nabla\psi_n$, even where $\nabla_h h$ is discontinuous, this property following from the absence of a term in $\nabla_h^2 h$. We note that the original approximation has not been changed, it has merely been reformulated. Thus, the jump conditions (2.5), which are inherent in the basic approximation, still hold, as can be confirmed using (3.2) and (3.7).

We may express the coefficients in (3.6) in terms of the functions Z_n, thereby eliminating W_n from the calculation altogether. Thus, for example, (3.2) and (3.7) imply that $\dot{\boldsymbol{W}} = S^T(\dot{\boldsymbol{Z}} - (\mathscr{A}^{-1}\mathscr{D})^T \boldsymbol{Z})$ and therefore

$$\dot{\boldsymbol{W}}\dot{\boldsymbol{W}}^T = \dot{S}^T\{\dot{\boldsymbol{Z}}\dot{\boldsymbol{Z}}^T - \dot{\boldsymbol{Z}}\boldsymbol{Z}^T(\mathscr{A}^{-1}\mathscr{D}) - (\mathscr{A}^{-1}\mathscr{D})^T \boldsymbol{Z}\dot{\boldsymbol{Z}}^T + (\mathscr{A}^{-1}\mathscr{D})^T \boldsymbol{Z}\boldsymbol{Z}^T(\mathscr{A}^{-1}\mathscr{D})\}S.$$

Integrating with respect to z from $-h$ to 0, we find that

$$\begin{aligned} C &= S^T\{\mathscr{C} - \mathscr{D}^T(\mathscr{A}^{-1}\mathscr{D}) - (\mathscr{A}^{-1}\mathscr{D})^T\mathscr{D} + (\mathscr{A}^{-1}\mathscr{D})^T\mathscr{A}(\mathscr{A}^{-1}\mathscr{D})\}S \\ &= S^T\{\mathscr{C} - \mathscr{D}^T\mathscr{A}^{-1}\mathscr{D}\}S, \end{aligned}$$

where

$$\mathscr{C}_{mn} = (\dot{Z}_m, \dot{Z}_n).$$

Combining this with two similar but more straightforward manipulations, we find the matrices A, B and C with $m-M+1, n-M+1$ components A_{mn}, B_{mn} and C_{mn}, respectively, can be written in the forms

$$A = A(h) = S^T \mathscr{A} S, \quad B = B(h) = S^T \mathscr{B} S, \quad C = C(h) = S^T\{\mathscr{C} - \mathscr{D}^T \mathscr{A}^{-1} \mathscr{D}\}S, \quad (3.8)$$

in which

$$\mathscr{B}_{mn} = (Z_m, Z_n'') + [Z_m Z_n']_{z=-h} \qquad (3.9)$$

has not previously been defined. We have tacitly assumed that the functions Z_n are chosen so that $\mathscr{A}$ is non-singular.

The final form of the approximation is therefore

$$\phi \approx \boldsymbol{\Psi}^T S^T(h) \boldsymbol{Z}, \qquad (3.10)$$

in which

$$\nabla_h \cdot A(h) \nabla_h \boldsymbol{\Psi} + \{B(h) - C(h)(\nabla_h h)^2\}\boldsymbol{\Psi} = \boldsymbol{0} \qquad (3.11)$$

determines $\boldsymbol{\Psi}$ and S is given by solving (3.7) subject to an initial condition of the form $S(\tilde{h}) = \tilde{S}$, where $\tilde{h}$ is a fixed reference depth.

We now comment on the cost involved in implementing the change of dependent variable implied by (3.2). The principal cost will be incurred in approximating a solution $S = S(h)$ to (3.7), but it should be noted that, for given M and N, the matrix S is a function only of Kh. S does not depend on the particular bed shape under consideration and it can be calculated and stored for a range of Kh in advance of the choice of topography h or the angular frequency σ.

The system of equations (3.11) has arisen from the change of dependent variables (3.2) and approximations to wave motion found via (3.11) will be the same as those found by Porter & Staziker (1995), assuming that the same functions Z_n are used. Some of the advantages of (3.11) over previous contributions will be discussed in what follows, but here we note one new opportunity. The classical mild-slope equation involves, in its derivation, the neglect of terms involving $(\nabla_h h)^2$ and $\nabla_h^2 h$, and a great deal of work has been done (for example by Kirby 1986; Massel 1993; Chamberlain & Porter 1995; Porter & Staziker 1995) in attempts to improve the approximation in cases where it produces unsatisfactory results. It is now understood that all of the deficiencies of the mild-slope equation in respect of mass-conservation and resonance by ripple beds are due to the neglect of the term in $\nabla_h^2 h$. Porter (2003) has derived a new mild-slope equation (given explicitly here in equation (3.21)) which does not neglect $\nabla_h^2 h$, and this equation has been shown to avoid the problems inherent in the classical mild-slope equation. The corresponding absence of $\nabla_h^2 h$ in (3.11) suggests that we also may neglect the term with $(\nabla_h h)^2$ in equation (3.11) so that we obtain the mild-slope approximation

$$\nabla_h \cdot A(h) \nabla_h \boldsymbol{\Psi} + B(h) \boldsymbol{\Psi} = 0 \qquad (3.12)$$

without losing mass conservation and one of the principal aims of this paper is to explore the validity of this new approximation which we do later by means of numerical experiments. We note here a useful consequence of removing $(\nabla_h h)^2$, that (3.12) applies everywhere and not just on subdomains where $\nabla_h h$ is continuous.

3.1. *Scale invariance property*

We recall that the functions Z_n are required to be real-valued and satisfy $Z_n' - K Z_n = 0$ on $z = 0$, but are otherwise arbitrary. One property of the matrices A, B and C in (3.11) is that they are unchanged if Z_n is replaced by $e_n(h) Z_n$, where $e_M(h), \ldots, e_N(h)$

are real-valued and non-zero. To show this, suppose that we use the circumflex accent to denote quantities occurring in (3.11) evaluated using $\hat{Z}_n = e_n(h)Z_n$ in place of Z_n and let $E = \mathrm{diag}(e_M, \ldots, e_N)$. It can easily be verified that

$$\hat{\mathscr{A}} = E\mathscr{A}E, \quad \hat{\mathscr{B}} = E\mathscr{B}E, \quad \hat{\mathscr{C}} = E\mathscr{C}E + \dot{E}\mathscr{D}E + E\mathscr{D}^T\dot{E} + \dot{E}\mathscr{A}E, \quad \hat{\mathscr{D}} = E\mathscr{D}E + E\mathscr{A}\dot{E}.$$

On using (3.7), it now follows that $\hat{\mathscr{A}}\hat{S}^{\bullet} + \hat{\mathscr{D}}\hat{S} = 0$, implying $\hat{S} = E^{-1}S$, and therefore $\hat{A} = \hat{S}^T\hat{\mathscr{A}}\hat{S} = S^T\mathscr{A}S = A$. Similarly $\hat{B} = B$ and, after some algebra, $\hat{C} = C$.

The practical advantage of this scale invariance property is that it avoids the need to normalize Z_n in any way and this simplifies the evaluation of the coefficients in (3.11). In contrast, the original version (2.4) of the system is sensitive to the way in which the members of the expansion set are scaled. In fact it can easily be shown that the coefficients in (2.4) have the scale invariance property only if the term in $\nabla_h^2 h$ is absent. In other words, we could have deduced (3.11) by seeking the scale invariant form of (2.4).

3.2. *Wave mode approximation*

Here we take $M = 0$ and use the functions $Z_0, Z_1, \ldots, Z_N$ defined by (2.6) and (2.8). However, the scale invariance property allows us to drop the factors $\mathrm{sech}(kh)$ and $\sec(k_n h)$ and use the simpler

$$Z_n(h,z) = \cos k_n(z+h) \quad (n = 0, \ldots, N), \tag{3.13}$$

in which $k_0 = -\mathrm{i}k$. This reduction in the dependence of Z_n on h, explicitly and through $k_n(h)$, means that $\dot{Z}_n$ is significantly simplified and the evaluation of the terms $\mathscr{C}_{mn}$ and $\mathscr{D}_{mn}$ correspondingly more straightforward.

As noted earlier, the functions Z_n are orthogonal at each value of h, and we easily find using (3.7) and (3.9) that

$$\mathscr{A}_{mn} = (4k_n)^{-1}\{2k_n h + \sin(2k_n h)\}\delta_{mn}, \quad \mathscr{B}_{mn} = -k_n^2\mathscr{A}_{nn}\delta_{mn}, \tag{3.14}$$

for $m, n = 0, \ldots, N$, where δ_{mn} is the Kronecker delta. The other inner products required to form the coefficients in (3.11), and relationships between them, are determined in the Appendix using a method that avoids laborious integrations. In particular, it is shown there that

$$k_n\dot{\mathscr{A}}_{nn} = 2k_n\mathscr{D}_{nn} - 2\dot{k}_n\mathscr{A}_{nn} \quad (n = 0, \ldots, N),$$

which can be rearranged as $(k_n\mathscr{A}_{nn}^{1/2})^{\bullet} = k_n\mathscr{A}_{nn}^{-1/2}\mathscr{D}_{nn}$, implying the matrix identity

$$(\mathscr{K}\mathscr{A}^{1/2})^{\bullet} = \mathscr{K}\mathscr{A}^{-1/2}\widetilde{\mathscr{D}}, \quad \mathscr{K} = \mathrm{diag}(k, k_1, \ldots, k_N), \quad \widetilde{\mathscr{D}} = \mathrm{diag}(\mathscr{D}_{00}, \ldots, \mathscr{D}_{NN}).$$

(We note that the square root of a diagonal matrix is found by taking the square root of each element.) This establishes an integrating factor for the diagonal elements of $\mathscr{A}$ and $\mathscr{D}$ in (3.7), for the latter may be written as

$$\mathscr{K}\mathscr{A}^{1/2}\dot{S} + (\mathscr{K}\mathscr{A}^{1/2})^{\bullet}S + \mathscr{K}\mathscr{A}^{-1/2}(\mathscr{D} - \widetilde{\mathscr{D}})S = 0$$

and therefore

$$\dot{P} + \mathscr{E}P = 0, \quad P = \mathscr{K}\mathscr{A}^{1/2}S. \tag{3.15}$$

Here $\mathscr{E} = \mathscr{K}\mathscr{A}^{-1/2}(\mathscr{D} - \widetilde{\mathscr{D}})\mathscr{A}^{-1/2}\mathscr{K}^{-1}$ is a real-valued matrix with diagonal elements equal to zero. Its off-diagonal elements may be deduced by referring to (3.14) and (A 4) and are

$$\left.\begin{aligned} \mathscr{E}_{n0} = \mathscr{E}_{0n} &= -4(kk_n)^{3/2}/\left\{u_0u_n\left(k^2 + k_n^2\right)\right\} \quad (n = 1, \ldots, N), \\ \mathscr{E}_{mn} = -\mathscr{E}_{nm} &= 4(k_mk_n)^{3/2}/\left\{u_nu_m\left(k_m^2 - k_n^2\right)\right\} \quad (m, n = 1, \ldots, N), \end{aligned}\right\} \tag{3.16}$$

in which

$$u_0 = \{2kh + \sinh(2kh)\}^{1/2}, \quad u_n = \{2k_n h + \sin(2k_n h)\}^{1/2} \quad (n = 1, \ldots, N).$$

The transformation to (3.15) is an immediate simplification of (3.7) for $N = 0$ and it determines S in that case since the scalar P must be a constant. The extension of the transformation to general N has been arranged so that all quantities remain real-valued. One consequence of this is the relationship

$$\mathscr{I}\mathscr{E}^T\mathscr{I} = -\mathscr{E}, \quad \mathscr{I} = \mathrm{diag}(1, -1, \ldots, -1), \tag{3.17}$$

where the matrix $\mathscr{I}$ arises as a result of including both propagating and evanescent modes within a single real-valued framework.

It follows from (3.15) and (3.17) that $\dot{P}^T\mathscr{I}P + P^T\mathscr{I}\dot{P} = 0$ and therefore that $P^T\mathscr{I}P$ is a constant matrix. By introducing the initial condition $P(\tilde{h}) = I$, where $\tilde{h}$ is a reference depth, the scaling matrix is given by

$$S = \mathscr{A}^{-1/2}\mathscr{K}^{-1}P, \quad P^T\mathscr{I}P = \mathscr{I}. \tag{3.18}$$

This structure has immediate consequences for (3.11). Referring to (3.8) and noting that $\mathscr{B} = \mathscr{I}\mathscr{K}^2\mathscr{A}$ by (3.14) we readily find that

$$A = P^T\mathscr{K}^{-2}P, \quad B = \mathscr{I},$$

which reveals the benefit of the transformation (3.15) for general N.

Applying the transformation (3.15) to the matrix C occurring in (3.11), we obtain

$$C = P^T\mathscr{G}P, \quad \mathscr{G} = \mathscr{K}^{-1}\mathscr{A}^{-1/2}\{\mathscr{C} - \mathscr{D}^T\mathscr{A}^{-1}\mathscr{D}\}\mathscr{A}^{-1/2}\mathscr{K}^{-1}.$$

The components of $\mathscr{G}$ can be evaluated by using the expressions given in the Appendix, noting in particular that

$$\{\mathscr{C} - \mathscr{D}^T\mathscr{A}^{-1}\mathscr{D}\}_{mn} = \mathscr{C}_{mn} - \sum_{p=0}^{N}\mathscr{A}_{pp}^{-1}\mathscr{D}_{pm}\mathscr{D}_{pn} \quad (m, n = 0, \ldots, N).$$

There is, however, an alternative version of $\mathscr{G}$ that is revealing. To derive this, we use the set $Z_0, Z_1, \ldots$ to form the expansion

$$\dot{Z}_m = \sum_{p=0}^{\infty}(Z_p, \dot{Z}_m)||Z_p||^{-2}Z_p,$$

which converges for $-h < z \leqslant 0$. Therefore

$$\mathscr{C}_{mn} = (\dot{Z}_m, \dot{Z}_n) = \sum_{p=0}^{\infty}(Z_p, \dot{Z}_m)(Z_p, \dot{Z}_n)||Z_p||^{-2} \quad (m, n = 0, \ldots, N)$$

and referring to the notation (3.7), we see that

$$\{\mathscr{C} - \mathscr{D}^T\mathscr{A}^{-1}\mathscr{D}\}_{mn} = \sum_{p=N+1}^{\infty}(Z_p, \dot{Z}_m)(Z_p, \dot{Z}_n)||Z_p||^{-2} \quad (m, n = 0, \ldots, N),$$

showing that C plays the part of an 'error term' in that it incorporates the modes that are omitted from the approximating set. Extending the notation (3.16) to $m, n \in \mathbb{N}$, we arrive at the compact expression

$$\mathscr{G}_{mn} = \sum_{p=N+1}^{\infty} k_p^{-2} \mathscr{E}_{pm} \mathscr{E}_{pn} \quad (m, n = 0, \ldots, N).$$

The final form of (3.11) resulting from the present choice of the approximating functions is

$$\nabla_h \cdot P^T \mathscr{K}^{-2} P \nabla_h \boldsymbol{\Psi} + \{\mathscr{I} - P^T \mathscr{G} P (\nabla_h h)^2\} \boldsymbol{\Psi} = \mathbf{0} \tag{3.19}$$

and, neglecting second-order terms in the bed slope, this may be approximated by

$$\nabla_h \cdot P^T \mathscr{K}^{-2} P \nabla_h \boldsymbol{\Psi} + \mathscr{I} \boldsymbol{\Psi} = \mathbf{0}, \tag{3.20}$$

which is the current counterpart of (3.12). The equations (3.19) and (3.20) are, respectively, the extensions to the multimode case of the equations derived in Porter (2003) and, in particular, the latter is the counterpart of the new mild-slope equation

$$\nabla_h \cdot k^{-2} \nabla_h \psi_0 + \psi_0 = 0, \tag{3.21}$$

derived in the earlier work. The role of the matrix P in coupling the propagating and evanescent modes ψ_n in the approximation (3.10) is clearly evident in (3.20). A justification for discarding the bed slope terms to obtain (3.20) will be given on the basis of numerical evidence later.

If h is constant, so are P, S, $\mathscr{K}$ and $\mathscr{A}$ and we may use (3.2) and (3.18) to recover

$$\nabla_h^2 \boldsymbol{\Phi} + \mathscr{K}^2 \mathscr{I} \boldsymbol{\Phi} = \mathbf{0}, \quad \boldsymbol{\Phi} = S \boldsymbol{\Psi}$$

from (3.11) and (3.20). Thus the functions in the approximation (2.3) satisfy the uncoupled equations

$$\nabla_h^2 \phi_0 + k^2 \phi_0 = 0, \quad \nabla_h^2 \phi_n - k_n^2 \phi_n = 0 \quad (n = 1, \ldots, N), \tag{3.22}$$

as expected.

3.3. *Coupled wave and bed mode approximation*

Here we modify the expansion set in accordance with the observation made by Athanassoulis & Belibassakis (1999) that the functions W_n in terms of which the approximation (3.3) is expressed satisfy the horizontal bed condition $W_n' = 0$ on $z = -h$. This is the case even though the functions defined in (3.13) satisfy the exact bed condition $Z_n' + \nabla_h h \cdot \nabla Z_n = 0$ on $z = -h$ (since all derivatives of $Z_0, \ldots, Z_N$ vanish on $z = -h$). We therefore extend the set (3.13) to include a 'sloping-bottom' mode Z_{-1} such that $Z_{-1}' \neq 0$ on $z = -h$.

Suppose then that we choose a real-valued function $b(h, z)$ having the properties $b'(h, 0) - K b(h, 0) = 0$ and $b'(h, -h) \neq 0$. We need to absorb the new mode into the notation that we have established and thus set $M = -1$ and define

$$Z_{-1}(h, z) = b(h, z) - \sum_{n=0}^{N} b_n Z_n(h, z) \;\; (-h \leqslant z \leqslant 0), \quad b_n = \mathscr{A}_{nn}^{-1}(b, Z_n), \tag{3.23}$$

which satisfies the free-surface condition $Z_{-1}' - K Z_{-1} = 0$ on $z = 0$, as our theory requires, and preserves the orthogonality of the expansion set. We therefore use the set $Z_{-1}, \ldots, Z_N$, where $Z_0, \ldots, Z_N$ are defined by (3.13) and Z_{-1} refers to the bed mode in the form (3.23). We note that the members of the resulting set $W_{-1}, \ldots, W_N$ are such that, in general, $W_n' \neq 0$ on $z = -h$, through the transformation (3.2).

The scale invariance property described earlier implies that the system (3.11) arising in the present application is insensitive to multiplication of b by any non-vanishing function of h. It is easily shown that

$$(b, Z_n) = -k_n^{-2} b'(h, -h) + O\left(k_n^{-3}\right),$$

for $n = 1, \ldots, N$ and that $k_n h = n\pi + O(n^{-1})$ for sufficiently large n and therefore $(b, Z_n) \sim n^{-2}$ for large n, whatever b.

The matrices $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ now have dimension $(N+2) \times (N+2)$, the elements $\mathscr{A}_{mn}$, $\mathscr{B}_{mn}$, $\mathscr{C}_{mn}$ and $\mathscr{D}_{mn}$ being unchanged from those given in the previous section for $m, n = 0, \ldots, N$. Only the new components corresponding to $m = -1$ and $n = -1$ have to be evaluated.

The choice (3.23) has some useful properties. Obviously, it ensures that the matrix $\mathscr{A}$ remains diagonal, the new component being

$$\mathscr{A}_{-1,-1} = \|b\|^2 - \sum_{n=0}^{N} \mathscr{A}_{nn}^{-1} (b, Z_n)^2.$$

Further, referring to (3.9), a straightforward calculation gives

$$\begin{aligned} \mathscr{B}_{-1,-1} &= K(b(h,0))^2 - \|b'\|^2 + \sum_{n=0}^{N} \mathscr{A}_{nn} (k_n b_n)^2, \\ \mathscr{B}_{-1,n} &= \mathscr{B}_{n,-1} = 0 \quad (n = 0, \ldots, N), \end{aligned}$$

and so $\mathscr{B}$ is also diagonal, as before. Therefore the system (3.10) reduces to $N+2$ decoupled equations for $\phi_{-1}, \ldots, \phi_N$ in the case where the depth h is constant and the equations (3.22) occurring in the earlier approximation also apply here, together with an additional equation for the new mode which is

$$\nabla_h^2 \phi_{-1} - k_{-1}^2 \phi_{-1} = 0, \quad k_{-1} = \{-\mathscr{B}_{-1,-1} / \mathscr{A}_{-1,-1}\}^{1/2}. \tag{3.24}$$

It can be shown that

$$k_{-1}^2 = \frac{\|b'\|^2 - K(b(h,0))^2 - \sum_{n=0}^{N} b_n^2 k_n^2 \mathscr{A}_{nn}}{\|b\|^2 - \sum_{n=0}^{N} b_n^2 \mathscr{A}_{nn}} = \frac{\sum_{N+1}^{\infty} b_n^2 k_n^2 \mathscr{A}_{nn}}{\sum_{N+1}^{\infty} b_n^2 \mathscr{A}_{nn}},$$

from which it is immediate that not only is k_{-1} real, but that $k_{-1} > k_{N+1}$. From this it is clear that the sloping-bottom mode contributes non-propagating effects in regions where the bed is flat. Furthermore, in regions where h is a constant, the sloping-bottom mode decays more rapidly than any of the wave modes that have been included in the approximation. This fact gives weight to the view that the sloping-bottom mode compensates for all the wave modes that have been omitted.

It should be noted here that this implementation of a sloping-bottom term differs from the one used by Athanassoulis & Belibassakis (1999). In the earlier work, the sloping-bottom mode was set to be identically zero in regions of constant depth. One of our aims in this paper is to show how allowing the bed mode to contribute a decaying term in such regions serves to compensate for the evanescent modes not included in the multi-mode expansion.

The remaining new terms required to form the coefficients in (3.11) are obtained as follows. First, recalling that $\mathscr{D}_{mn} = (Z_m, \dot{Z}_n)$ and using $(Z_{-1}, Z_n) = 0$ for $n = 0, \ldots, N$

we readily find that

$$\mathscr{D}_{-1,-1} = (b, \dot{b}) - \sum_{n=0}^{N} b_n\{(b, \dot{Z}_n) + (\dot{b}, Z_n)\} + \sum_{m=0}^{N}\sum_{n=0}^{N} b_m b_n \mathscr{D}_{mn},$$

$$\mathscr{D}_{-1,n} = (b, \dot{Z}_n) - \sum_{m=0}^{N} b_m \mathscr{D}_{mn} \quad (n = 0, \dots, N),$$

$$\mathscr{D}_{n,-1} = -\mathscr{D}_{-1,n} - [Z_{-1}]_{z=-h}.$$

Finally,

$$\mathscr{C}_{-1,-1} = \|\dot{b}\|^2 - \sum_{n=0}^{N} \left\{2b_n(\dot{b}, \dot{Z}_n) + 2\dot{b}_n(\dot{b}, Z_n) - \mathscr{A}_{nn}\dot{b}_n^2\right\},$$

$$+ \sum_{m=0}^{N}\sum_{n=0}^{N} b_n\{b_m \mathscr{C}_{mn} + 2\dot{b}_m \mathscr{D}_{mn}\},$$

$$\mathscr{C}_{-1,n} = \mathscr{C}_{n,-1} = (\dot{b}, \dot{Z}_n) - \sum_{m=0}^{N}\{b_m \mathscr{C}_{mn} + \dot{b}_m \mathscr{D}_{mn}\} \quad (n = 0, \dots, N).$$

We note that, with h fixed,

$$b(h, z) = \sum_{n=0}^{\infty} b_n Z_n,$$

the sum converging to b at each $z \in (-h, 0]$, and therefore

$$Z_{-1}(h, z) = \sum_{n=N+1}^{\infty} b_n Z_n(h, z),$$

again with pointwise convergence for $z \in (-h, 0]$. The significance of the mode Z_{-1} thus reduces with increasing N, as expected. However, the purpose of the mode is to ensure good approximations for small values of N, as recognized by Athanassoulis & Belibassakis (1999) and confirmed below.

It remains to choose b. One possibility is to use the Athanassoulis & Belibassakis (1999) bed mode function $b(h, z) = z^3 + hz^2$ (on appealing to scale-invariance to remove a factor of h^2 from the expression in equation (2.10)), but the linear function

$$b(h, z) = 1 + Kz$$

satisfies the assumed conditions, has the merit of simplicity and is such that

$$b'' \equiv 0, \quad \dot{b} \equiv 0, \quad b'(-h, h) = K, \quad \|b\|^2 = \{1 - (1 - Kh)^3\}/3K,$$

$$b_n = -K\big/\big(k_n^2 \mathscr{A}_{nn}\big), \quad \dot{b}_n = 2K\mathscr{D}_{nn}\big/\big(k_n^2 \mathscr{A}_{nn}^2\big), \quad (b, \dot{Z}_n) = b_n - (1 - Kh),$$

$\dot{b}_n$ having been determined by using formulae given in the Appendix. These expressions enable the new components defined above to be evaluated using the quantities arising in the earlier approximation and therefore the computational cost of using the extended expansion set is virtually negligible.

We note that the counterpart of the reduction of (3.11) to (3.19) can be contrived for the extended approximating set, but it is a comparatively unwieldy construction.

4. Numerical experiments

Our aim in this section is to examine numerical results with particular regard to the importance of the bed slope term and the validity of the new approximation (3.12). For this purpose we follow previous authors by considering a class of two-dimensional scattering problems in which the varying topography is confined to a bounded interval. Thus, the motion is assumed to be independent of y and the bedform such that $h(x)$ is continuous, has a piecewise continuous derivative and satisfies

$$h(x) = \begin{cases} h_0 & (x < 0), \\ h_1 & (x > \ell), \end{cases}$$

where $\ell > 0$, h_0 and h_1 being constants.

It is efficient to describe the formulation of the scattering problem with the bed mode included. Specializing to the wave mode only approximation and to the equation (3.19) is a simple matter. Referring to (3.22) and (3.24) we take the approximation (3.1) to ϕ in the forms

$$\varphi(x,z) = \left\{A^{(0)}\mathrm{e}^{\mathrm{i}k^{(0)}x} + B^{(0)}\mathrm{e}^{-\mathrm{i}k^{(0)}x}\right\}Z_0(h_0,z) + \sum_{n=1}^{N} B_n^{(0)}\mathrm{e}^{k_n^{(0)}x}Z_n(h_0,z) + B_{-1}^{(0)}\mathrm{e}^{k_{-1}^{(0)}x}Z_{-1}(h_0,z) \qquad (x<0) \quad (4.1)$$

and

$$\varphi(x,z) = \left\{A^{(1)}\mathrm{e}^{\mathrm{i}k^{(1)}(\ell-x)} + B^{(1)}\mathrm{e}^{-\mathrm{i}k^{(1)}(\ell-x)}\right\}Z_0(h_1,z) + \sum_{n=1}^{N} B_n^{(1)}\mathrm{e}^{k_n^{(1)}(\ell-x)}Z_n(h_1,z) + B_{-1}^{(1)}\mathrm{e}^{k_{-1}^{(1)}(\ell-x)}Z_{-1}(h_1,z) \qquad (x>\ell). \quad (4.2)$$

Here $k^{(i)}$ and $k_n^{(i)}$ refer to the roots of (2.7) and (2.9), respectively, with $h = h_i$, and the superscripts attached to the bed mode exponent k_{-1} (defined in (3.24)) have the corresponding meaning.

The amplitudes $A^{(0)}$ and $A^{(1)}$ of the incident propagating waves from the left and right, respectively, are considered to be assigned and the remaining amplitudes are to be determined as part of the solution process. The exact solution of the problem in these two domains is given by setting $B_{-1}^{(0)} = B_{-1}^{(1)} = 0$ and letting $N \to \infty$. In the approximation, however, we retain $B_{-1}^{(0)}$ and $B_{-1}^{(1)}$ and determine them along with the other evanescent mode amplitudes as part of the solution process. The bed mode terms in the approximations (4.1) and (4.2) can therefore be regarded as compensating for the truncation of the full infinite set of evanescent modes.

This is an appropriate point to note that Athanassoulis & Belibassakis (1999) used the sloping-bottom mode in a different way. By devising boundary conditions for the mode at $x = 0$ and $x = \ell$, they confined its effect to the domain on which the topography is varying. Although it has no direct role to play outside the interval $[0, \ell]$ in terms of improving the approximation to the bed condition, we allow it to continue smoothly into the flat-bed regions where it contributes to the solution in the form of a 'truncation mode', as we noted above.

Boundary conditions for the solution in $0 < x < \ell$ can be deduced from (4.1) and (4.2). Referring to the notation of (3.1), extended to include the bed mode, we have

$$\boldsymbol{\Phi}(0-) = (0, A^{(0)}, 0, \ldots, 0)^T + \left(B_{-1}^{(0)}, B^{(0)}, B_1^{(0)}, \ldots, B_N^{(0)}\right)^T \equiv \boldsymbol{A}_0 + \boldsymbol{B}_0,$$

$$\boldsymbol{\Phi}(\ell+) = (0, A^{(1)}, 0, \ldots, 0)^T + \left(B_{-1}^{(1)}, B^{(1)}, B_1^{(1)}, \ldots, B_N^{(1)}\right)^T \equiv \boldsymbol{A}_1 + \boldsymbol{B}_1,$$

$$\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}x}(0-) = -\boldsymbol{K}^{(0)}\{\boldsymbol{A}_0 - \boldsymbol{B}_0\}, \quad \frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}x}(\ell+) = \boldsymbol{K}^{(1)}\{\boldsymbol{A}_1 - \boldsymbol{B}_1\},$$

where $\boldsymbol{K}^{(i)} = \mathrm{diag}(k_{-1}^{(i)}, -\mathrm{i}k^{(i)}, k_1^{(i)}, \ldots, k_N^{(i)})$ is an $(N+2)\times(N+2)$ matrix, since $M = -1$. If we now use the transformation $\boldsymbol{\Phi} = S(h)\boldsymbol{\Psi}$, noting that $\mathrm{d}h/\mathrm{d}x = 0$ for $x < 0$ and $x > \ell$, and the continuity of the elements of $\boldsymbol{\Psi}$ and their derivatives we obtain

$$\frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}x}(0+) - \boldsymbol{K}^{(0)}\boldsymbol{\Psi}(0+) = -2S^{-1}(h_0)\boldsymbol{K}^{(0)}\boldsymbol{A}_0, \quad \frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}x}(\ell-) + \boldsymbol{K}^{(1)}\boldsymbol{\Psi}(\ell-) = 2S^{-1}(h_1)\boldsymbol{K}^{(1)}\boldsymbol{A}_1, \tag{4.3}$$

which are the required boundary conditions, and

$$\frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}x}(0+) + \boldsymbol{K}^{(0)}\boldsymbol{\Psi}(0+) = 2S^{-1}(h_0)\boldsymbol{K}^{(0)}\boldsymbol{B}_0, \quad \frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}x}(\ell-) - \boldsymbol{K}^{(1)}\boldsymbol{\Psi}(\ell-) = -2S^{-1}(h_1)\boldsymbol{K}^{(1)}\boldsymbol{B}_1, \tag{4.4}$$

which recover $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ once the solution for $\boldsymbol{\Psi}$ in $0 < x < \ell$ is known.

To complete the structural part of the solution process, we write (3.11) as the first-order system

$$\boldsymbol{X}' = \begin{pmatrix} 0 & A^{-1}(h) \\ C(h)\left(\dfrac{\mathrm{d}h}{\mathrm{d}x}\right)^2 - B(h) & 0 \end{pmatrix}\boldsymbol{X}, \quad \boldsymbol{X} = \begin{pmatrix} \boldsymbol{\Psi} \\ A(h)\dfrac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}x} \end{pmatrix}, \tag{4.5}$$

where $\boldsymbol{X}$ is vector of length $2(N+2)$, (4.3) as

$$Q_0^-\boldsymbol{X}(0+) = -2S^{-1}(h_0)\boldsymbol{K}^{(0)}\boldsymbol{A}_0, \quad Q_1^+\boldsymbol{X}(\ell-) = -2S^{-1}(h_1)\boldsymbol{K}^{(1)}\boldsymbol{A}_1, \tag{4.6}$$

and (4.4) as

$$Q_0^+\boldsymbol{X}(0+) = -2S^{-1}(h_0)\boldsymbol{K}^{(0)}\boldsymbol{B}_0, \quad Q_1^-\boldsymbol{X}(\ell-) = -2S^{-1}(h_1)\boldsymbol{K}^{(1)}\boldsymbol{B}_1, \tag{4.7}$$

in which

$$Q_i^{\pm} = (\pm\boldsymbol{K}^{(i)} \;\; A^{-1}(h_i))$$

is an $(N+2)\times 2(N+2)$ matrix.

Finally, if $\boldsymbol{X}^{(j)}$ $(j = 1, \ldots, 2(N+2))$ denote linearly independent solutions of (4.5), obtained by solving initial-value problems, then the general solution is

$$\boldsymbol{X}(x) = \boldsymbol{X}_s(x)\boldsymbol{C}, \quad \boldsymbol{X}_s = (\boldsymbol{X}^{(1)}, \ldots, \boldsymbol{X}^{2(N+2)})$$

where $\boldsymbol{C}$ is a constant vector of length $2(N+2)$. Using (4.6) and (4.7) in turn, we obtain

$$\begin{pmatrix} Q_0^-\boldsymbol{X}_s(0) \\ Q_1^+\boldsymbol{X}_s(\ell) \end{pmatrix}\boldsymbol{C} = 2\begin{pmatrix} -S^{-1}(h_0)\boldsymbol{K}^{(0)} & 0 \\ 0 & S^{-1}(h_1)\boldsymbol{K}^{(1)} \end{pmatrix}\begin{pmatrix} \boldsymbol{A}_0 \\ \boldsymbol{A}_1 \end{pmatrix}$$

and

$$\begin{pmatrix} Q_0^+\boldsymbol{X}_s(0) \\ Q_1^-\boldsymbol{X}_s(\ell) \end{pmatrix}\boldsymbol{C} = 2\begin{pmatrix} S^{-1}(h_0)\boldsymbol{K}^{(0)} & 0 \\ 0 & -S^{-1}(h_1)\boldsymbol{K}^{(1)} \end{pmatrix}\begin{pmatrix} \boldsymbol{B}_0 \\ \boldsymbol{B}_1 \end{pmatrix}.$$

Eliminating $\boldsymbol{C}$ gives a relationship between $(\boldsymbol{A}_0 \;\, \boldsymbol{A}_1)^T$ and $(\boldsymbol{B}_0 \;\, \boldsymbol{B}_1)^T$, which includes

$$\begin{pmatrix} B^{(0)} \\ B^{(1)} \end{pmatrix} = \mathscr{S}\begin{pmatrix} A^{(0)} \\ A^{(1)} \end{pmatrix}, \quad \mathscr{S} = \begin{pmatrix} R_0 & T_1 \\ T_0 & R_1 \end{pmatrix},$$

where $\mathscr{S}$ is the standard scattering matrix. Thus R_0 (R_1) and T_0 (T_1) are the complex amplitudes of the reflected and transmitted waves generated by a wave of unit amplitude from the left (right).

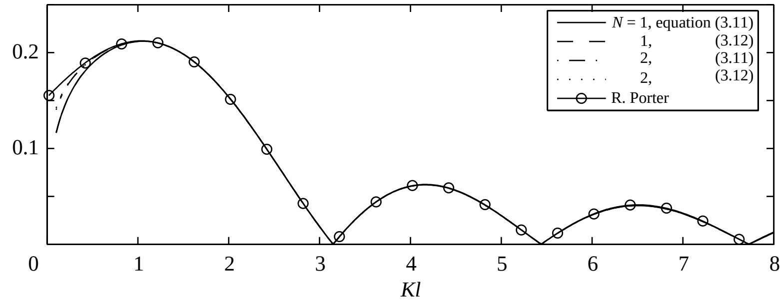

FIGURE 1. Reflected amplitudes computed with the inclusion of a sloping-bottom mode, that is $M = -1$. Results are given for $N = 1$ and $N = 2$ for each of the approximations (3.11) and (3.12). The solid line with circles is included for comparison and shows the 'full linear' solution computed by R. Porter using the method of Porter & Porter (2000).

We remark here on the determination of the transformation matrix S. In the first example below, Kh takes values between 0.5 and 1 and in the second illustration $0.81 < Kh < 2.45$, so S is required only in the interval $0.5 \leqslant Kh < 2.45$ in what follows. As noted previously, an initial condition is required to determine S and we choose to set it equal to the identity matrix at some value of Kh, using a simple numerical method to approximate it for the remaining required values of Kh. A key point to be emphasized is that, for fixed M and N, the same S may be used in both of the following illustrations. Furthermore, the values of S determined could also be used for yet more bed shapes, including those involving three-dimensional topography, see, for example, Belibassakis, Athanassoulis & Gerostathis (2001), provided that Kh is in the range mentioned above.

4.1. *A symmetric elevation*

We consider the test problem of a symmetric elevation in an otherwise flat bed by taking

$$Kh(x) = 2(x/\ell)^2 - 2(x/\ell) + 1$$

for $0 \leqslant x \leqslant \ell$. This corresponds to a hump whose maximum height occupies half the still-water depth and the test problem thus defined has been considered previously by Porter & Staziker (1995).

In figure 1 we show approximations to $|R_0|$ for an extended range of $K\ell$, and examine the effect both of varying N from 1 to 2 and of making the mild-slope approximation used to derive equation (3.12) from (3.11). For most values of $K\ell$ all five versions of $|R_0|$ are indistinguishable to the eye. For small values of $K\ell$ some differences are apparent, but even here the effect of making the mild-slope approximation is not visible to the eye, for the solid and broken lines lie almost on top of each other and the dotted and 'dash-dot' lines are also near-identical. We note that small $K\ell$ corresponds to steep bed slopes and in this case larger values of N are required to resolve the solution.

In order that finer distinctions may be made, the next two figures are concerned with the reflected amplitude for a more restricted range of $K\ell$ than was the case in figure 1. Figures 2 and 3 show $|R_0|$ plotted against $K\ell$, with attention focused on the largest local maximum of $|R_0|$.

Figure 2 shows approximations to the reflected amplitude computed without the inclusion of a sloping-bottom mode. Solid curves with accents were found via numerical solutions of (3.11), and broken lines imply that (3.12) has been used instead.

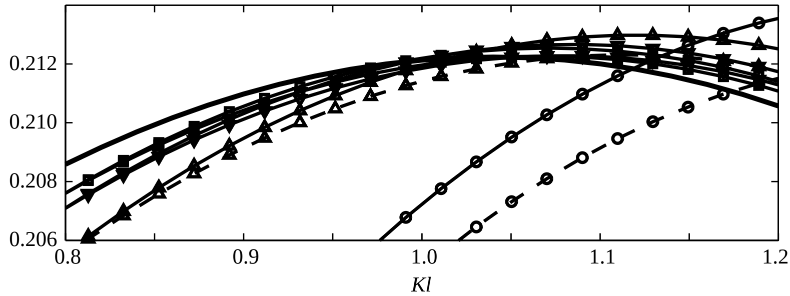

FIGURE 2. Reflected amplitudes computed without the inclusion of a sloping-bottom mode. A solid line with ○, 1 evanescent mode; △, 5 evanescent modes; ∇, 10 evanescent modes; □, 15 evanescent modes. Broken lines indicate the effect of removing $C(h)$ from (3.11). The solid line with no symbols is included for comparison and shows the 'full linear' solution computed by R. Porter using the method of Porter & Porter (2000).

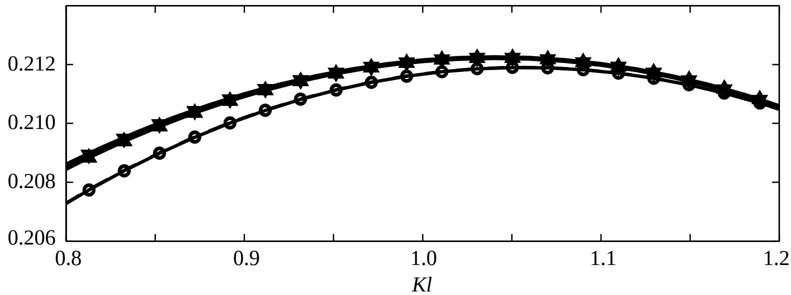

FIGURE 3. Reflected amplitudes computed with the inclusion of a sloping-bottom mode. A solid line with ○, 1 evanescent mode; △, 2 evanescent modes; ∇, 3 evanescent modes. Broken lines indicate the effect of removing $C(h)$ from (3.11). The solid line with no symbols shows the 'full linear' solution computed by R. Porter using the method of Porter & Porter (2000). The excellent agreement of certain results means that not all curves described here are discernible to the eye.

The symbols indicate how many evanescent modes were used as follows: circles – 1 evanescent mode, upward pointing triangles – 5 evanescent modes, downward pointing triangles – 10 evanescent modes and squares – 15 evanescent modes. The solid curve with no symbols was computed using the method of Porter & Porter (2000) and shows the solution obtained from a solution of the full problem (2.1). We note in passing that the current example was not considered in Porter & Porter (2000), but that R. Porter has used the method described in that paper for our comparison. Convergence of the current method is evident, and it is notable that the mild-slope process of removing the term in $(\nabla_h h)^2$ is least detrimental when the approximation is most accurate. In other words, we have found that removing the term involving $(\nabla_h h)^2$ only affects results by an amount of the same order as the error already present in (3.11).

Figure 3 is similar to figure 2 except that solutions to (3.11) and (3.12) were obtained with the inclusion of the sloping-bottom mode, with $b(z) = 1 + Kz$. Fewer evanescent modes were necessary when compared with the results considered in figure 2. Circles again correspond to the inclusion of 1 evanescent mode, but now the upward and downward pointing triangles correspond to 2 and 3 evanescent modes, respectively. Broken lines, found via (3.11) with $C(h)$ omitted, are hard to see, for they lie almost directly on their solid line counterparts.

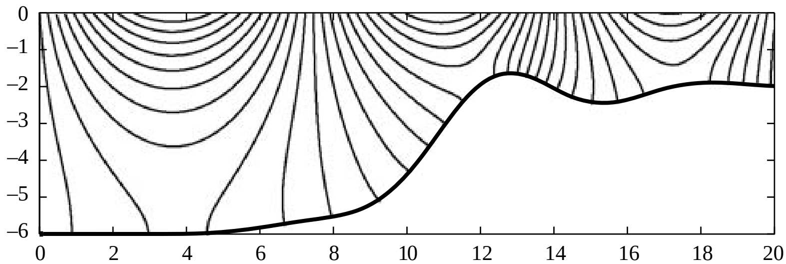

FIGURE 4. Vertical cross-section showing contours of Im(φ) in the case where $M = -1$ and $N = 1$ for the Athanassoulis & Belibassakis test problem.

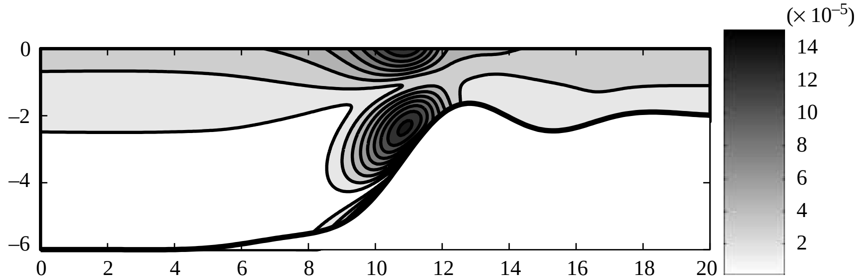

FIGURE 5. Vertical cross-section showing the absolute value of the change in φ when $C(h)$ is removed from (3.11). Here $N = 1$ and $M = -1$.

4.2. *An underwater shoal*

Athanassoulis & Belibassakis (1999) considered the following case of an underwater shoal with bottom corrugations. We take

$$h = \tfrac{1}{2}\Big(h_0 + h_1 + (h_1 - h_0)\tanh\Big(3\pi\Big(\frac{x}{\ell} - \tfrac{1}{2}\Big)\Big)\Big) - 0.67\exp\left(-\tfrac{1}{20}(x - 12.5)^2\right)\cos(x - 12.5),$$

an expression which corrects a sign error in equation (6.3) of Athanassoulis & Belibassakis. For ease of comparison with the earlier paper we give the various parameters in dimensional form: $h_0 = 6\,\mathrm{m}$, $h_1 = 2\,\mathrm{m}$, $\ell = 20\,\mathrm{m}$, $\sigma = 2\,\mathrm{s}^{-1}$ so that $K = 0.41\,\mathrm{m}^{-1}$, to 2 decimal places.

Our results are in excellent agreement with those of Athanassoulis & Belibassakis, and contours indistinguishable from those in their figure 7 are obtained with only one evanescent mode, when the sloping-bottom mode is also included. Figure 4 shows contours of Im(φ) and should be compared with figure 7(a) of Athanassoulis & Belibassakis. In accordance with Athanassoulis & Belibassakis, we have not labelled our contours, but note that Im(φ) varies between -6.5 and 6.9, to 1 decimal place. Figure 4 was computed using equation (3.11), but, as discussed in the next paragraph, there is no change discernable to the eye if (3.12) is used instead. (There is similar excellent agreement between Re(φ) and figure 7(b) of Athanassoulis & Belibassakis.)

Figure 5 corresponds to the inclusion of one evanescent mode and the sloping-bottom mode and shows the effect of removing $C(h)$ from (3.11). Contours are shown of the absolute value of the change in φ, the approximation to ϕ, resulting in the removal of $C(h)$. We see that, when the approximation is close to the true solution, the removal of $C(h)$ results in only a very small change in φ.

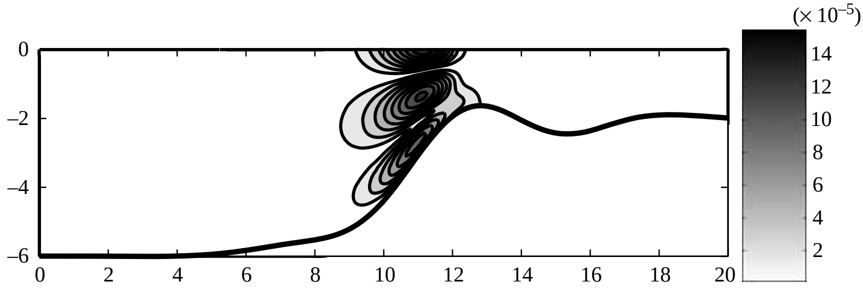

FIGURE 6. As figure 5, but here $N = 2$.

Figure 6 is as figure 5, but with $N = 2$, and we see that with the (guaranteed) improvement in results afforded by an increase in N, the effect of removing $C(h)$ from (3.11) is reduced.

5. Conclusions

The multi-mode variational approximation to wave scattering over topography, previously derived by Massel (1993) and Porter & Staziker (1995), has been re-examined and the differential equation system that determines the approximation simplified by a transformation of the dependent variables. One beneficial effect of the new system is that the expansion set used in the approximation is scale invariant and it turns out that the transformation used, albeit for a different reason, is the only one leading to this property. Scale invariance is practically significant as it considerably reduces the effort required to evaluate the coefficients arising in the system. Reference to Silva, Salles & Palacio (2002) shows the level of complexity that can arise in calculations of this type without the benefit of scale invariance and the use of an indirect evaluation strategy such as that devised here.

It is important to recall that the matrix S which defines the transformation is a function of Kh and can therefore be calculated and stored before a specific bed shape or angular wave frequency have been decided upon. It follows that the transformation is no more expensive to apply in fully three-dimensional problems than in the test problems considered in this paper. Belibassakis *et al.* (2001) have given results derived from a multi-mode decomposition in a three dimensional problem, and it is likely that results found via our equation (3.12) will be very similar to the results that they obtained. We note at this point that Belibassakis & Athanassoulis (2004) have also used a multi-mode expansion with a bed-sloping mode in another three-dimensional context, to determine a Green's function, in the case where the topography consists of parallel contours. The three-dimensional nature of the problem in this case arises from the point-source nature of the Green's function, rather than from the topographical features.

The transformation of the system does not change the approximation, it simply recasts it in a simpler and more convenient form. However, the new system (3.11) does suggests a further simplification, obtained by discarding the square of the bed slope $(\nabla_h h)^2$. The net effect of these various steps is the differential equation system given in (3.12), which may be contrasted with the original system (2.4). Since (3.12) is independent of $\nabla_h h$ it applies where the bed slope is discontinuous, removing the need that arises in the original form of the approximation to assemble the solution in a piecemeal fashion and applying jump conditions. Numerical experiments, a selection of which are described earlier, suggest that the final reduction to (3.12) does not increase the order of the error in the approximation.

We have also examined the inclusion of a sloping-bottom mode, for which the system (3.12) still applies when the approximation is reduced to its simplest form. By making an appropriate choice of the new mode, no additional terms have to be calculated to evaluate the extended approximation. Although this step was motivated by the work of Athanassoulis & Belibassakis (1999), who improved on the approximations of Massel (1993) and Porter & Staziker (1995), we have used the sloping-bottom mode in a different way by allowing its effect to penetrate regions where the bed is flat. There it has the effect of a 'truncation term', compensating for the modes that are excluded from the approximation.

We have found in numerical calculations that the scattering properties of demanding bedforms are determined accurately by a three-mode approximation: the sloping-bottom mode; the propagating wave mode; and a single evanescent mode. Overall, the approximation that we have derived provides a practical and relatively straightforward way of estimating linear scattering over uneven beds. We have found that the optimal approach is to incorporate a sloping-bottom mode together with the mild-slope reduction to equation (3.12); the neglect of $(\nabla_h h)^2$ incurs least error when the approximation is most accurate and the inclusion of a sloping-bottom mode improves accuracy. It seems unlikely that a simpler method can be formulated that has the same accuracy.

Finally, we comment that (3.12) is likely to be attractive to coastal engineers, for there is no requirement to know (or approximate) derivatives of h. This fact is significant, for example, if h has to be interpolated from field data.

The authors are grateful to Dr Richard Porter, of the University of Bristol, for supplying results used for comparison in figures 1, 2 and 3.

Appendix

Our objective here is to evaluate the matrix elements required in (3.7) and (3.11) with Z_n given by (3.13). We recall that

$$\mathscr{A}_{mn} = \mathscr{A}_{nn}\delta_{mn}, \quad \mathscr{D}_{mn} = (Z_m, \dot{Z}_n), \quad \mathscr{C}_{mn} = (\dot{Z}_m, \dot{Z}_n),$$

Z_n being defined in (3.13); $\mathscr{A}_{nn}$ is given in (3.14).

Now since $Z_n'' + k_n^2 Z_n = 0$ and $k_n = k_n(h)$, we have

$$\dot{Z}_n'' + k_n^2 \dot{Z}_n + 2k_n \dot{k}_n Z_n = 0, \tag{A 1}$$

and we note that

$$[Z_n]_{z=-h} = 1, \quad [\dot{Z}_n]_{z=-h} = 0, \quad [\dot{Z}_n']_{z=-h} = -k_n^2. \tag{A 2}$$

Using these values and the fact that Z_n and $\dot{Z}_n$ satisfy the free-surface condition, we obtain

$$(Z_m'', \dot{Z}_n) - (Z_m, \dot{Z}_n'') = [Z_m' \dot{Z}_n - Z_m \dot{Z}_n']_{z=-h}^{z=0} = -k_n^2,$$

which with (A 1) gives the identity

$$\left(k_m^2 - k_n^2\right)(Z_m, \dot{Z}_n) - 2k_n \dot{k}_n (Z_m, Z_n) = k_n^2. \tag{A 3}$$

We therefore have

$$\mathscr{D}_{mn} = k_n^2 \big/ \left(k_m^2 - k_n^2\right) \quad (m \neq n). \tag{A 4}$$

With $m = n$, (A 3) shows that

$$2\dot{k}_n \mathscr{A}_{nn} = -k_n, \tag{A 5}$$

giving an expression for $\dot{k}_n$. Further, differentiating $||Z_n||^2$, we see that

$$\dot{\mathscr{A}}_{nn} = 2\mathscr{D}_{nn} + 1, \tag{A 6}$$

which combines with (A 5) to yield

$$k_n \dot{\mathscr{A}}_{nn} = 2k_n \mathscr{D}_{nn} - 2\dot{k}_n \mathscr{A}_{nn}. \tag{A 7}$$

By direct calculation,

$$2\dot{\mathscr{A}}_{nn} = 1 + \sin(2k_n h)\{1 + \cos(2k_n h)\}/4k_n \mathscr{A}_{nn}, \tag{A 8}$$

and (A 6) now gives

$$\mathscr{D}_{nn} = \{\sin(4k_n h) - 4k_n h\}/32k_n \mathscr{A}_{nn}. \tag{A 9}$$

To evaluate the elements of $\mathscr{C}$, we follow the same strategy and first use (A 2) to give

$$(\dot{Z}''_m, \dot{Z}_n) - (\dot{Z}_m, \dot{Z}''_n) = [\dot{Z}'_m \dot{Z}_n - \dot{Z}_m \dot{Z}'_n]_{z=-h}^{z=0} = 0.$$

Then (A 1) implies that

$$\left(k_m^2 - k_n^2\right)\mathscr{C}_{mn} + 2k_m \dot{k}_m \mathscr{D}_{mn} - 2k_n \dot{k}_n \mathscr{D}_{nm} = 0$$

whence

$$\mathscr{C}_{mn} = (k_m k_n)^2 \left(k_m^2 - k_n^2\right)^{-2} \left\{\mathscr{A}_{mm}^{-1} + \mathscr{A}_{nn}^{-1}\right\} \quad (m \neq n). \tag{A 10}$$

A similar procedure gives $\mathscr{C}_{nn}$. Evaluating $(\dot{Z}''_n, \ddot{Z}_n) - (\dot{Z}_n, \ddot{Z}''_n)$, combining the result with the derivative of (A 6) and using preceding formulae, we find that

$$6\mathscr{C}_{nn} = 2k_n^2 \mathscr{A}_{nn} + \{3\sin(4k_n h) - 4\sin^3(2k_n h) - 12k_n h\}/32k_n \mathscr{A}_{nn}^2. \tag{A 11}$$

REFERENCES

Athanassoulis, G. A. & Belibassakis, K. A. 1999 A consistent coupled-mode theory for the propagation of small-amplitude water waves over variable bathymetry regions. *J. Fluid Mech.* **389**, 275–301.

Belibassakis, K. A. & Athanassoulis, G. A. 2004 Three-dimensional Green's function of harmonic water waves over a bottom topography with different depths at infinity. *J. Fluid Mech.* **510**, 267–302.

Belibassakis, K. A., Athanassoulis, G. A. & Gerostathis, Th. 2001 A coupled-mode model for the refraction–diffraction of linear waves over steep three-dimensional bathymetry. *Appl. Ocean Res.* **23**, 319–336.

Berkhoff, J. C. W. 1972 Computation of combined refraction–diffraction. *Proc. 13th Conf. on Coastal Engng, July 1972, Vancouver, Canada*, vol. 2, pp. 471–490. ASCE.

Berkhoff, J. C. W. 1976. Mathematical models for simple harmonic linear waves. Wave diffraction and refraction. *Delft Hydr. Rep.* W 154-IV.

Chamberlain, P. G. & Porter, D. 1995 The modified mild-slope equation. *J. Fluid Mech.* **291**, 393–407.

Chamberlain, P. G. & Porter, D. 1996 Approximations to wave trapping by topography. *J. Fluid Mech.* **325**, 357–476.

Chamberlain, P. G. & Porter, D. 1999 Scattering and near-trapping of water waves by axisymmetric topography. *J. Fluid Mech.* **388**, 335–354.

Chamberlain, P. G. & Porter, D. 2005 Wave scattering in a two-layer fluid of varying depth. *J. Fluid Mech.* **524**, 207–228.

Kirby, J. T. 1986 A general wave equation for waves over rippled beds. *J. Fluid Mech.* **162**, 171–186.

MASSEL, S. R. 1993 Extended refraction–diffraction equation for surface waves. *Coastal Engng* **19**, 97–126.

PORTER, D. 2003 The mild-slope equations. *J. Fluid Mech.* **494**, 51–63.

PORTER, D. & STAZIKER, D. J. 1995 Extensions of the mild-slope equation. *J. Fluid Mech.* **300**, 367–382.

PORTER, R. & PORTER, D. 2000 Water wave scattering by a step of arbitrary profile. *J. Fluid Mech.* **411**, 131–164.

SILVA, R., SALLES, P. & PALACIO, A. 2002 Linear waves propagating over a rapidly varying finite porous bed. *Coastal Engng* **44**, 239–260.

SMITH, R. & SPRINKS, T. 1975 Scattering of surface waves by a conical island. *J. Fluid Mech.* **72**, 373–384.